Game Theory - Repeated Games

Stéphane

today :-}
Outline

1 Basic Game Theoretic Concept
   - Basic Concepts
   - Properties
   - Equilibrium concepts

2 Repeated Game
Outline

1. Basic Game Theoretic Concept
   - Basic Concepts
   - Properties
   - Equilibrium concepts

2. Repeated Game
What is a normal form game?

**Definition**

An *n*-player game can be represented by a mapping

\[ R : A_1 \times A_2 \times \ldots \times A_n \mapsto \mathbb{R}^n \]

where \( A_i \) denotes the discrete set of action available to player \( i \)

- \( a = (a_1, a_2, \ldots, a_n) \) is the joint action of the players
- \( R(a) \) is the payoff for each player (\( R_i(a) \) is the payoff of the \( i^{th} \) player, i.e. the \( i^{th} \) component of \( R(a) \))

For a 2-player game, \( R \) can be represented by 2 matrices.
Basic Concepts

What is a strategy?

Definition
A **pure strategy** is a synonym for an action \( a \in A_i \).

Definition
A **mixed strategy** \( \pi_i \) is a probability distribution over the action space \( A_i \).
Basic Game Theoretic Concept

Repeated Game

Basic Concepts

examples

**Example (Battle of the sexes)**

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**Problem:** Where to go on a date: Soccer or Opera?

**Requirements:**
1. avoid to be alone
2. be at the best place

**Example (Prisoners’ dilemma)**

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- Problem: me and my buddy got busted!
- Cooperate: I shut my mouth
- Defect: I blame my buddy
# Game Theory is a big field

### other concepts

- **simultaneous or sequential**: play simultaneously: each player makes a decision in turn (game tree).
- **perfect/imperfect information**: ability to observe the actions of the opponent(s)
- **complete/incomplete information**: complete information: knowledge of the structure of the games (payoffs matrices).
- **one stage/multistage game**: the outcome of a joint action can be a new game
Game Theory is a big field

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**Properties of the payoffs**

**stochastic game:** payoff can be stochastic

**Bayesian game:** incomplete information game: at the start of the game, some player have private information that others do not (example: bargaining game)

**constant/general sum game:** for each joint action \( a \in \prod_i A_i \), the sum of the payoff \( \sum_i R_i(a) \) can be constant.

**Team Game or Cooperative game:** all the players receive the same payoff for a joint action.
Dominance

**Definition**

An outcome $X$ **strongly dominates** another outcome $B$ if all agents receive a higher utility in $X$ compared to $Y$.

$$a > b \iff \forall i \in [1..n] R_i(a) > R_i(b)$$

An outcome $X$ **weakly dominates** (or simply dominates) another outcome $B$ if at least one agent receives a higher utility in $X$ and no agent receives a lesser utility compared to outcome $Y$.

$$a \geq b \iff \exists j | R_j(a) > R_j(b) \text{ and } \forall i \in [1..n], i \neq j R_i(a) \geq R_i(b)$$
Properties

Pareto Optimality

Definition

A Pareto optimal outcome is one such that there is no other outcome where some players can increase their payoffs without decreasing the payoff of other players. A non-dominated outcome is Pareto optimal.
Regret measures how much worse an algorithm performs to the best static strategy.

**Definition**

the **external regret** is the difference that a player would receive if it were to play the pure strategy $j$ instead of playing according to $\pi$.

**Definition**

the **internal regret** is the benefit that player $i$ would get by switching all of its plays of action $j$ to action $k$ instead.

**Definition**

the **total internal (external) regret** is the max of the internal (external) regret.
Definition

An equilibrium is a self-reinforcing distribution over strategy profile.

- Assumption: players are rational (issue with bounded rationality)
- Different natures of equilibrium.
Minimax equilibrium for constant-sum games

minimize the payoff of the opponent: If deviation from equilibrium, the opponent gets an advantage.

Minimax value of a game for player 1

\[ \min_{y} \max_{x} R_1(x, y) \]

Properties

- There exists at least one minimax equilibrium in constant sum game.
- set of minimax equilibrium is convex, all have the same value
Nash equilibrium: rationality

**mutual best response**
if the strategy of the opponent remains fixed, the player does not benefit by changing its strategy

**Properties**
- **existence:**
  - pure strategy Nash equilibrium may not always exist
  - but there always exists a mixed strategy Nash equilibrium
- **complexity to find a Nash equilibrium:** there exists exponential time algorithms to compute it, but nobody proved it is NP-Complete.
### Examples

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Basic Game Theoretic Concept

Equilibrium concepts

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**Nash equilibrium**  D,C and C,D and one mixed strategy $(\frac{3}{4}, \frac{1}{4})$

**Pareto Optimal**  D,C and C,D

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**Nash equilibrium** \((D, D)\) is the only Nash equilibria of the game.

**Pareto Optimal** \((D, C), (C, D)\) and \((C, C)\)

**N.B.** A Nash equilibrium may not be Pareto Optimal
Equilibrium concepts

**Correlated equilibrium**

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- both agents play mixed strategy \((\frac{1}{2}, \frac{1}{2})\): average payoff is 2.5
- how to avoid bad outcome?
Correlated equilibrium

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Correlated equilibrium

Players can observe a public random variable and make their decision based on that observation. Player’s distribution may no longer be independent. solved by linear program
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- flip a (fair?) coin
- head: husband cooperates
- tail: wife cooperates

**Example (Traffic light)**

- 2 actions: Stop or Go
- model the light as being randomly Green or Red. It is the public random variable
- choose life
Outline

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   - Basic Concepts
   - Properties
   - Equilibrium concepts

2. Repeated Game
Outline

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2. Repeated Game
Repeated Game

Definition
In the repeated game a game $M$ (called stage game) is played over and over again

- one shot game: there is no tomorrow
- repeated game: model a likelihood of playing the game again with the same opponent
- finitely/infinitely repeated game
What is a strategy in a repeated game?

Example:
Tit for Tat strategy
- Play the action played by the opponent the last round
- Tit for tat strategy can be an equilibrium strategy in PD or Chicken.
What is a strategy in a repeated game?

In the repeated game, a pure strategy depends also on the history of play thus far.

Example

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Tit for Tat strategy
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Payoff criterion

**Average criterion**
Average payoff received throughout the game by player $i$:

$$\sum_{t=0}^{\infty} M_i(a^t)$$

where $a^t$ is the joint action of iteration $t$.

**Discounted-sum criterion**
Discounted sum of the payoff received throughout the game by player $i$:

$$\sum_{t=0}^{\infty} \gamma^t M_i(a^t)$$
Basic Game Theoretic Concept

### Payoff Space for a two-player game

- **$n \times n$** two-player game
- $R$ and $C$ are the matrices of the row and column player.
- $V = \{(R(i, j), C(j, i))|(i, j) \in [1..n]^2\}$
- the **payoff space** is the Convex Hull $\mathcal{H}$ with vertices in $V$

#### Proof.

$\forall (x, y) \in \mathcal{H}, \exists \lambda \in \mathbb{R}^{n^2} \mid x = \sum_{i=1}^{n} \lambda_i R(i) \text{ and } y = \sum_{i=1}^{n} \lambda_i C(j)$

with $\sum_{i=1}^{n} \lambda_i = 1$.

Play the joint action $i$ with the proportion $\lambda_i$. 

□
Example and payoff with independent distribution

Battle of the Sexes

Prisoners’ dilemma
Minimax Value

Feasible region for equilibrium

Minimax value for row and column player:

\[ v_r = \min_y \max_x R(x, y) \]
\[ v_c = \min_x \max_y C(x, y) \]

The minimax value security value
It defines a feasible region (for an equilibrium)

\[ \mathcal{F} = \{(x, y) \in \mathcal{H} | x \geq v_r, y \geq v_c \}. \]
Feasible region for Battle of Sexes and Prisoners’ dilemma

Pareto frontier

Payoff of woman

Payoff of man

Payoff of column player

Payoff of row player
Folk Theorem

**Theorem**

Any payoff \( r \in F \) can be sustained by a Nash equilibrium.

**Proof.**

Build strategies that converge to the desired payoff and that make it non-rational to deviate from the strategy.
Learning in Games

Desirable Properties

**Convergence:** a learning algorithm should converge

**Rationality:** play optimally against a stationary opponent

**no regret:** avoid regrets

Or are they?

Is it possible to find equilibrium that can be good for both players?
"That's all Folks!"